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NORMAL FUZZY PROBABILITY FOR GENERALIZED QUADRATIC FUZZY SETS

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Abstract. A generalized quadratic fuzzy set is a generalization of a quadratic fuzzy number. Zadeh defines the probability of the fuzzy event using the probability. We define the normal fuzzy probability on $\mathbb{R}$ using the normal distribution. And we calculate the normal fuzzy probability for generalized quadratic fuzzy sets.

1. Introduction

We define the generalized quadratic fuzzy set and calculate four operations of two generalized triangular fuzzy sets([3]). Four operations are based on the Zadeh's extension principle([5]). And Zadeh defines the probability of fuzzy event as follows.

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, where Ω denotes the sample space, $\mathfrak{F}$ the $\sigma-$algebra on Ω, and P a probability measure. A fuzzy set A on Ω is called a fuzzy event. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([6]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \qquad \mu_A(\omega) : \Omega \to [0, 1].$$

We defined the normal fuzzy probability using the normal distribution and calculated the normal fuzzy probability for quadratic fuzzy number([2]). And then we had the explicit formula for the normal fuzzy probability for trigonometric fuzzy number.

In this paper, we calculate the normal fuzzy probability for generalized quadratic fuzzy sets.

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2. Preliminaries

Let $(\Omega, \mathfrak{F}, P)$ be a probability space, and X be a random variable defined on it. Let g be a real-valued Borel-measurable function on $\mathbb{R}$. Then $g(X)$ is also a random variable.

Definition 2.1. We say that the mathematical expectation of $g(X)$ exists if

$$E[g(X)] = \int_\Omega g(X(\omega))\, dP(\omega) = \int_\Omega g(X)\, dP$$

is finite.

We note that a random variable X defined on $(\Omega, \mathfrak{F}, P)$ induces a measure P_X on a Borel set $B \in \mathfrak{B}$ defined by the relation $P_X(B) = P\{X^{-1}(B)\}$. Then P_X becomes a probability measure on $\mathfrak{B}$ and is called the probability distribution of X. It is known that if $E[g(X)]$ exists, then g is also integrable over $\mathbb{R}$ with respect to P_X. Moreover, the relation

$$\int_\Omega g(X)dP = \int_{\mathbb{R}} g(t)dP_X(t) \tag{2.1}$$

holds. We note that the integral on the right-hand side of (2.1) is the Lebesgue- Stieltjes integral of g with respect to P_X. In particular, if g is continuous on $\mathbb{R}$ and $E[g(X)]$ exists, we can rewrite (2.1) as follows

$$\int_\Omega g(X)dP = \int_{\mathbb{R}} g dP_X = \int_{-\infty}^{\infty} g(x)dF(x),$$

where F is the distribution function corresponding to P_X, and the last integral is a Riemann-Stieltjes integral.

Let F be absolutely continuous on $\mathbb{R}$ with probability density function $f(x) = F'(x)$. Then $E[g(X)]$ exists if and only if the integral $\int_{-\infty}^{\infty} |g(x)|f(x)dx$ is finite and in that case we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)dF(x).$$

Example 2.2. *Let the random variable X have the normal distribution given by the probability density function*

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(x-m)^2}{2\sigma^2}\Big), \quad x \in \mathbb{R},$$

where $\sigma^2 > 0$ *and* $m \in \mathbb{R}$. *Then* $E[|X|^\gamma] < \infty$ *for every* $\gamma > 0$, *and we have*

$$E[X] = m \quad \text{and} \quad E[(X-m)^2] = \sigma^2.$$

The induced measure P_X *is called the normal distribution.*

A fuzzy set A on Ω is called a *fuzzy event*. Let $\mu_A(\cdot)$ be the membership function of the fuzzy event A. Then the probability of the fuzzy event A is defined by Zadeh([6]) as

$$\widetilde{P}(A) = \int_\Omega \mu_A(\omega)\, dP(\omega), \quad \mu_A(\omega) : \Omega \to [0,1].$$

Definition 2.3. The normal fuzzy probability $\widetilde{P}(A)$ of a fuzzy set A on $\mathbb{R}$ is defined by

$$\widetilde{P}(A) = \int_{\mathbb{R}} \mu_A(x)\, dP_X,$$

where P_X is the normal distribution.

Definition 2.4. A quadratic fuzzy number is a fuzzy number A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < \alpha, \ \ \beta \le x, \\ a(x-k)^2 + 1, & \alpha \le x < \beta. \end{cases}$$

The above quadratic fuzzy number is denoted by $A = [\alpha, k, \beta]$.

Applying the extension principle to algebraic operations for fuzzy sets, we have the following definitions for extended operations.

Definition 2.5. The addition, subtraction, multiplication, and division of two fuzzy numbers are defined as

1. Addition $A(+)B$:

$$\mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

2. Subtraction $A(-)B$:

$$\mu_{A(-)B}(z) = \sup_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

3. Multiplication $A(\cdot)B$:

$$\mu_{A(\cdot)B}(z) = \sup_{z=x\cdot y} \min\{\mu_A(x), \mu_B(y)\}, \ x \in A, y \in B.$$

4. Division $A(/)B$:

$$\mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\},\ x \in A, y \in B.$$

3. Generalized quadratic fuzzy sets

In this section, we define the generalized quadratic fuzzy set and apply the extended algebraic operations to these fuzzy sets. A generalized quadratic fuzzy set is symmetric and may not have value 1.

Definition 3.1. A fuzzy set A with a membership function

$$\mu_A(x) = \begin{cases} 0, & x < x_1,\ \ x_2 \le x, \\ -a(x-x_1)(x-x_2) = -a(x-m)^2 + p, & x_1 \le x < x_2, \end{cases}$$

where $0 < a$ and $0 < p \le 1$ is called a *generalized quadratic fuzzy set* and denoted by $[[a, x_1, x_2]]$ or $[[a, m, p]]_+$.

Theorem 3.2. ([3]) *Let $A = [[a, x_1, x_2]] = [[a, m, p]]_+$ and $B = [[b, x_3, x_4]] = [[b, n, q]]_+$ be generalized quadratic fuzzy sets. Suppose $p \le q$ and $\mu_B(x) \ge p$ on $[k_1, k_2]$. Then we have the followings.*

(1) $A(+)B$ is a fuzzy set with a membership function

$$\mu_{A(+)B}(x) = \begin{cases} 0, & x < x_1 + x_3,\ x_2 + x_4 \le x, \\ f_1(x), & x_1 + x_3 \le x < m + k_1, \\ p, & m + k_1 \le x < m + k_2, \\ f_2(x), & m + k_2 \le x < x_2 + x_4, \end{cases}$$

where

$$\begin{aligned} f_1(x) = \frac{1}{a^2 - 2ab + b^2}\Big(& a^2 q + b^2 p - ab(p+q) - abm(a + b + an + bn) \\ & - abn(am + bm + an + bn) + 2ab(am + bm + an + bn)x \\ & - ab(a+b)x^2 + 2ab(m + n - x) \cdot \sqrt{g(x)}\Big) \end{aligned}$$

and

$$\begin{aligned} f_2(x) = \frac{1}{a^2 - 2ab + b^2}\Big(& a^2 q + b^2 p - ab(p+q) - abm(a + b + an + bn) \\ & - abn(am + bm + an + bn) + 2ab(am + bm + an + bn)x \\ & - ab(a+b)x^2 - 2ab(m + n - x) \cdot \sqrt{g(x)}\Big), \end{aligned}$$

where $g(x) = ab(m+n)^2 + (a-b)(p-q) - 2ab(m+n)x + abx^2$.

(2) $A(-)B$ is a fuzzy set with a membership function

$$\mu_{A(-)B}(x) = \begin{cases} 0, & x < x_1 - x_4, x_2 - x_3 \le x, \\ f_1(x), & x_1 - x_4 \le x < m - k_2, \\ p, & m - k_2 \le x < m - k_1, \\ f_2(x), & m - k_1 \le x < x_2 - x_3, \end{cases}$$

where

$$\begin{aligned} f_1(x) = \frac{1}{a^2 - 2ab + b^2}\Big(& a^2q + b^2p - ab(p+q) - ab^2x^2 \\ & - abm(am + bm - an - bn) - abn(an + bn - am - bm) \\ & + 2ab(am + bm - an - bn)x + 2ab(m - n - x) \cdot \sqrt{g(x)}\Big) \end{aligned}$$

and

$$\begin{aligned} f_2(x) = \frac{1}{a^2 - 2ab + b^2}\Big(& a^2q + b^2p - ab(p+q) - ab^2x^2 \\ & - abm(am + bm - an - bn) - abn(an + bn - am - bm) \\ & + 2ab(am + bm - an - bn)x - 2ab(m - n - x) \cdot \sqrt{g(x)}\Big), \end{aligned}$$

where $g(x) = ab(m-n)^2 + (a-b)(p-q) - 2ab(m-n)x + abx^2$.

(3) If $p = q$, $A(\cdot)B$ is a fuzzy set with a membership function

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0, & x < x_1x_3, \ \ x_2x_4 \le x, \\ f(x), & x_1x_3 \le x < x_2x_4, \end{cases}$$

where

$$f(x) = \frac{1}{2}(-am^2 - bn^2 + 2p) - \sqrt{ab}x + \frac{1}{2}\sqrt{g(x)}$$

and

$$\begin{aligned} g(x) = & -am^2(am^2 + 3bn^2) - bn^2(bn^2 + 3am^2) + 2(am^2 + bn^2 - 2p)^2 \\ & + 8p(am^2 + bn^2 - p) + 8abmnx - \frac{1}{8\sqrt{abx}}\Big\{-8(am^2 + bn^2 - 2p)^3 \\ & + 8(am^2 + bn^2 - 2p)h_1(x) - 16h_2(x)\Big\}, \end{aligned}$$

and where

$$\begin{aligned} h_1(x) = & \, am^2(am^2 + 2bn^2) + bn^2(bn^2 + 2am^2) \\ & - 6p(am^2 + bn^2 - p) - 4abmnx - 2abx^2 \end{aligned}$$

and

$$h_2(x) = abm^2n^2(am^2+bn^2-4p) - am^2p(am^2-3p) - bn^2p(bn^2-3p) \\ - 2p^3 - 2abmn(am^2+bn^2-2p)x + ab(am^2+bn^2+2p)x^2.$$

(4) $A(/)B$ is a fuzzy set with a membership function

$$\mu_{A(/)B}(x) = \begin{cases} 0, & x < x_1/x_4, \quad x_2/x_3 \le x, \\ f_1(x), & x_1/x_4 \le x < m/k_2, \\ p, & m/k_2 \le x < m/k_1, \\ f_2(x), & m/k_1 \le x < x_2/x_3, \end{cases}$$

where

$$f_1(x) = \frac{1}{b^2 - 2abx^2 + a^2x^4}\Big(2a^2bmnx^3 + 2ab^2mnx - b^2(am^2+p) \\ - ab(am^2+bn^2+p+q)x^2 - a^2(bn^2-q)x^4 \\ + 2abx(m-nx)\cdot\sqrt{g(x)}\Big)$$

and

$$f_2(x) = \frac{1}{b^2 - 2abx^2 + a^2x^4}\Big(2a^2bmnx^3 + 2ab^2mnx - b^2(am^2+p) \\ - ab(am^2+bn^2+p+q)x^2 - a^2(bn^2-q)x^4 \\ - 2abx(m-nx)\cdot\sqrt{g(x)}\Big),$$

and where $g(x) = b(am^2-p+q) - 2abmnx + a(bn^2+p-q)x^2$.

4. Normal fuzzy probability for generalized quadratic fuzzy sets

In this section, we derive the explicit formula for the normal fuzzy probability for generalized quadratic fuzzy sets and give some examples.

THEOREM 4.1. *Let $X \sim N(m,\sigma^2)$ and $A = [[a,b,c]]$ be generalized quadratic fuzzy set. Then the normal fuzzy probability of a generalized quadratic fuzzy set A is*

$$\widetilde{P}(A) = \frac{a\sigma}{\sqrt{2\pi}}(m-b)\exp\Big(-\frac{(c-m)^2}{2\sigma^2}\Big) - \frac{a\sigma}{\sqrt{2\pi}}(m-c)\exp\Big(-\frac{(b-m)^2}{2\sigma^2}\Big) \\ - a(\sigma^2+m^2-m(b+c)+bc)\Big(F_N(\frac{c-m}{\sigma}) - F_N(\frac{b-m}{\sigma})\Big),$$

where $F_N(\alpha)$ is the standard normal distribution, that is,

$$F_N(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp\Big(-\frac{t^2}{2}\Big) dt.$$

Proof. Since

$$\mu_A(x) = \begin{cases} 0, & x < b, \;\; c \le x, \\ -a(x-b)(x-c), & b \le x < c, \end{cases}$$

where $0 < a$, we have

$$\begin{aligned} \widetilde{P}(A) &= \int_{\mathbb{R}} \mu_A(x)\; dP_X \\ &= \frac{-1}{\sqrt{2\pi}\sigma} \int_b^c a(x-b)(x-c) \exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big) dx. \end{aligned}$$

Putting $\dfrac{x-m}{\sigma} = t$, then

$$\begin{aligned} \widetilde{P}(A) &= \frac{-a}{\sqrt{2\pi}} \int_{\frac{b-m}{\sigma}}^{\frac{c-m}{\sigma}} \Big((\sigma t + m)^2 - (b+c)(\sigma t + m) + bc\Big) \exp\Big(-\frac{t^2}{2}\Big) dt \\ &= \frac{-a\sigma^2}{\sqrt{2\pi}} \int_{\frac{b-m}{\sigma}}^{\frac{c-m}{\sigma}} t^2 \exp\Big(-\frac{t^2}{2}\Big) dt \\ &\quad - \frac{a\sigma(2m-b-c)}{\sqrt{2\pi}} \int_{\frac{b-m}{\sigma}}^{\frac{c-m}{\sigma}} t \exp\Big(-\frac{t^2}{2}\Big) dt \\ &\quad - \frac{a(m^2 - m(b+c) + bc)}{\sqrt{2\pi}} \int_{\frac{b-m}{\sigma}}^{\frac{c-m}{\sigma}} \exp\Big(\frac{-t^2}{2}\Big) dt \\ &= \frac{a\sigma}{\sqrt{2\pi}} (\frac{c-m}{\sigma}\sigma + 2m - b - c) \exp\Big(-\frac{(c-m)^2}{2\sigma^2}\Big) \\ &\quad - \frac{a\sigma}{\sqrt{2\pi}} (\frac{b-m}{\sigma}\sigma + 2m - b - c) \exp\Big(-\frac{(b-m)^2}{2\sigma^2}\Big) \\ &\quad - a(\sigma^2 + m^2 - m(b+c) + bc)\Big(F_N(\frac{c-m}{\sigma}) - F_N(\frac{b-m}{\sigma})\Big) \\ &= \frac{a\sigma}{\sqrt{2\pi}} (m-b) \exp\Big(-\frac{(c-m)^2}{2\sigma^2}\Big) - \frac{a\sigma}{\sqrt{2\pi}} (m-c) \exp\Big(-\frac{(b-m)^2}{2\sigma^2}\Big) \\ &\quad - a(\sigma^2 + m^2 - m(b+c) + bc)\Big(F_N(\frac{c-m}{\sigma}) - F_N(\frac{b-m}{\sigma})\Big). \end{aligned}$$

Thus the proof is complete. □

EXAMPLE 4.2. *1. Let* $A = [[\frac{2}{27}, 2, 8]] = [[\frac{2}{27}, 5, \frac{2}{3}]]_+$ *be a generalized quadratic fuzzy number. Then the normal fuzzy probability of* A *with respect to* $X \sim N(3, 2^2)$ *is* 0.2126.

In fact, putting $\frac{x-3}{2} = t$*, we have*

$$
\begin{aligned}
\widetilde{P}(A) &= \frac{-2}{27\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{5}{2}} \Big((2t+3)^2 - 10(2t+3) + 16\Big) \exp\Big(-\frac{t^2}{2}\Big) dt \\
&= \frac{-8}{27\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{5}{2}} t^2 \exp\Big(-\frac{t^2}{2}\Big) dt + \frac{16}{27\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{5}{2}} t \exp\Big(-\frac{t^2}{2}\Big) dt \\
&\quad + \frac{10}{27\sqrt{2\pi}} \int_{-\frac{1}{2}}^{\frac{5}{2}} \exp\Big(\frac{-t^2}{2}\Big) dt \\
&= \frac{4}{27\sqrt{2\pi}} \exp\Big(-\frac{25}{8}\Big) + \frac{20}{27\sqrt{2\pi}} \exp\Big(-\frac{1}{8}\Big) \\
&\quad + \frac{2}{27}\Big(F_N(\frac{5}{2}) - F_N(-\frac{1}{2})\Big) \\
&= 0.2126.
\end{aligned}
$$

2. Let $B = [[\frac{3}{64}, 3, 11]] = [[\frac{3}{64}, 7, \frac{3}{4}]]_+$ *be a generalized quadratic fuzzy number. Then the normal fuzzy probability of* A *with respect to* $X \sim N(5, 3^2)$ *is* 0.1828. *In fact, putting* $\frac{x-5}{3} = t$*, we have*

$$
\begin{aligned}
\widetilde{P}(A) &= \frac{-3}{64\sqrt{2\pi}} \int_{-\frac{2}{3}}^{2} \Big((3t+5)^2 - 14(3t+5) + 33\Big) \exp\Big(-\frac{t^2}{2}\Big) dt \\
&= \frac{-27}{64\sqrt{2\pi}} \int_{-\frac{2}{3}}^{2} t^2 \exp\Big(-\frac{t^2}{2}\Big) dt + \frac{9}{16\sqrt{2\pi}} \int_{-\frac{2}{3}}^{2} t \exp\Big(-\frac{t^2}{2}\Big) dt \\
&\quad + \frac{9}{16\sqrt{2\pi}} \int_{-\frac{2}{3}}^{2} \exp\Big(\frac{-t^2}{2}\Big) dt \\
&= \frac{9}{32\sqrt{2\pi}} \exp(-2) + \frac{27}{32\sqrt{2\pi}} \exp\Big(-\frac{2}{9}\Big) \\
&\quad + \frac{9}{64}\Big(F_N(2) - F_N(-\frac{2}{3})\Big) \\
&= 0.1828.
\end{aligned}
$$

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